Game Theory, Spring 2024
Lecture # 3

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1 Auctions with private values

There is a single object for sale, and I potential bidders. Bidder i assigns value V_i to the object. V_i is distributed on $[0, 1]$ according to F, independently and identically across bidders. F has a continuous density f and full support. Bidder i knows her own value, but does not know the values of her competitors.

2 First-price sealed-bid auctions

In a first-price sealed-bid auction, the highest bidder wins and pays the amount she bid. We can formally define it as follows:

Definition 1 (**First-price sealed-bid auction**). *A first-price sealed-bid auction is a Bayesian game that consists of the following:*

1. *Players:* $\{\text{Bidder } 1, \dots, \text{Bidder } I\}$,
2. *Actions:* $A_1 = \cdots = A_I = \mathbb{R}_+$,
3. *Types:* $\Theta_1 = \cdots = \Theta_I = [0, 1]$,
4. *Probability distribution over type profiles:*

$$\mathbb{P}\big[V_1 \leq v_1, \dots, V_I \leq v_I\big] = F(v_1) \times \cdots \times F(v_I),$$

5. *Payoffs:*

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j\neq i} b_j, \\ \frac{1}{\#win}(v_i - b_i) & \text{if } b_i = \max_{j\neq i} b_j, \\ 0 & \text{otherwise,} \end{cases}$$

where #win is the number of winners in the auction.

We are going to look at symmetric Bayesian Nash equilibria of this game in pure strategies. A pure strategy is $\beta : [0,1] \to \mathbb{R}_+$, mapping valuations to bids. Suppose β is strictly increasing, continuosly differentiable, and $\beta(0) = 0$. Suppose bidder i has valuation v_i and bids b_i. The expected utility of bidder i is then given by:

$$\mathbb{P}\big[\text{win with } b_i \text{ against } \beta\big](v_i - b_i).$$

The winning probabilty $\mathbb{P}\big[\text{win with } b_i \text{ against } \beta\big]$ is equal to:

$$\begin{aligned}
&\mathbb{P}\big[b_i \geq \beta(V_1), \dots, b_i \geq \beta(V_{i-1}), b_i \geq \beta(V_{i+1}), \dots b_i \geq \beta(V_I)\big] \\
&= \mathbb{P}\big[\beta^{-1}(b_i) \geq V_1, \dots, \beta^{-1}(b_i) \geq V_{i-1}, \beta^{-1}(b_i) \geq V_{i+1}, \dots \beta^{-1}(b_i) \geq V_I\big] \\
&= \mathbb{P}\big[V_1 \leq \beta^{-1}(b_i)\big] \times \dots \times \mathbb{P}\big[V_{i-1} \leq \beta^{-1}(b_i)\big] \times \mathbb{P}\big[V_{i+1} \leq \beta^{-1}(b_i)\big] \times \dots \times \mathbb{P}\big[V_I \leq \beta^{-1}(b_i)\big] \\
&= \underbrace{F\big(\beta^{-1}(b_i)\big) \times \dots \times F\big(\beta^{-1}(b_i)\big) \times F\big(\beta^{-1}(b_i)\big) \times \dots \times F\big(\beta^{-1}(b_i)\big)}_{I-1 \text{ times}} \\
&= \big[F\big(\beta^{-1}(b_i)\big)\big]^{I-1}
\end{aligned}$$

Define $G(x) \equiv \big[F\big(x\big)\big]^{I-1}$, and let $g(x) \equiv G'(x)$. We can then write down the expected utility of bidder i as follows:

$$G\big(\beta^{-1}(b_i)\big)(v_i - b_i).$$

Taking the first-order condition with respect to b_i, we get:

$$g\big(\beta^{-1}(b_i)\big)\big[\beta^{-1}\big]'(b_i)(v_i - b_i) - G\big(\beta^{-1}(b_i)\big) = 0.$$

In equilibrium, we must have $b_i = \beta(v_i)$, hence we get:

$$\begin{aligned}
& g(v_i)\frac{1}{\beta'(v_i)}\big(v_i - \beta(v_i)\big) - G(v_i) = 0 \\
\Leftrightarrow\ & g(v_i)v_i = \beta(v_i)g(v_i) + \beta'(v_i)G(v_i) \\
\Leftrightarrow\ & g(v_i)v_i = \underbrace{\beta(v_i)G'(v_i) + \beta'(v_i)G(v_i)}_{\text{Product rule}} \\
\Leftrightarrow\ & g(v_i)v_i = \big[\beta(v_i)G(v_i)\big]'.
\end{aligned}$$

We can therefore write:

$$\int_0^{v_i} g(x)x dx = \int_0^{v_i} \big[\beta(x)G(x)\big]' dx = \beta(v_i)G(v_i) - \underbrace{\beta(0)G(0)}_{=0} = \beta(v_i)G(v_i).$$

We now have our equilibrium candidate

$$\beta(v_i) = \frac{1}{G(v_i)} \int_0^{v_i} xg(x)dx.$$

Recall that $g(x) = G'(x) = \frac{\partial}{\partial x}\big[F\big(x\big)\big]^{I-1} = (I-1)\big[F\big(x\big)\big]^{I-2} f(x)$, hence we can rewrite $\beta(v_i)$ as follows:

$$\beta(v_i) = \frac{1}{\big[F\big(x\big)\big]^{I-1}} \int_0^{v_i} x(I-1)\big[F\big(x\big)\big]^{I-2} f(x)dx.$$

Example 1. *Suppose V_i is uniformly distributed on $[0,1]$ for each i, we then have $F(x) = x$ and $f(x) = 1$. The equilibrium bidding strategy is then given by:*

$$\beta(v_i) = \frac{1}{v_i^{I-1}} \int_0^{v_i} x(I-1)x^{I-2}1dx = \frac{I-1}{I}v_i.$$

Revenue achieved by the seller in equilibrium is given by $R^* = \max\{\beta(V_1), \ldots, \beta(V_I)\}$. Let $V^{(1)} \equiv \max\{V_1, \ldots, V_I\}$. The cdf of $V^{(1)}$ is given by:

$$H(x) \equiv \mathbb{P}\left[V^{(1)} \leq x\right] = \mathbb{P}\left[V_1 \leq x, \ldots, V_I \leq x\right] = \big[F(x)\big]^I$$

The density of $V^{(1)}$ is then $h(x) = H'(x) = I\big[F(x)\big]^{I-1} f(x)$. Expected revenue can then be written as:

$$\mathbb{E}\, R^* = \int\limits_0^1 \beta(x) I\big[F(x)\big]^{I-1} f(x) dx.$$

In the uniform case, we get

$$\mathbb{E}\, R^* = \int\limits_0^1 \frac{I-1}{I} x I x^{I-1} 1 dx = \frac{I-1}{I+1}.$$