Game Theory, Spring 2024

Lecture # 5

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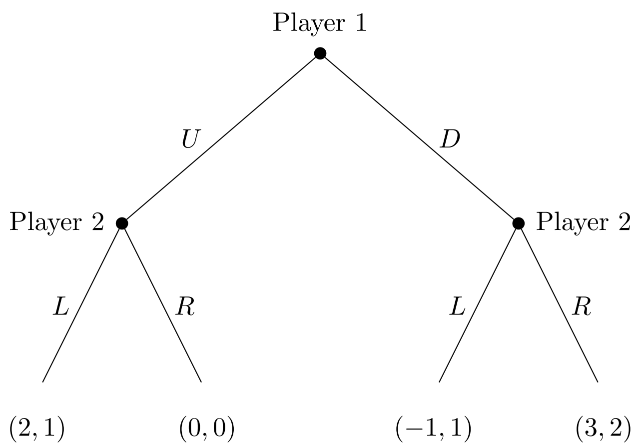
1 Extensive-form games: examples

1.1 Perfect information without exogenous uncertainty

Example 1. *Consider the following extensive-form game:*



To formally define the extensive-form game in Example 1, we need to specify the set of players, the set of histories of play, specifying which player moves at each non-terminal history, and the payoffs achieved by the players at each terminal history. The formal definition is as follows:

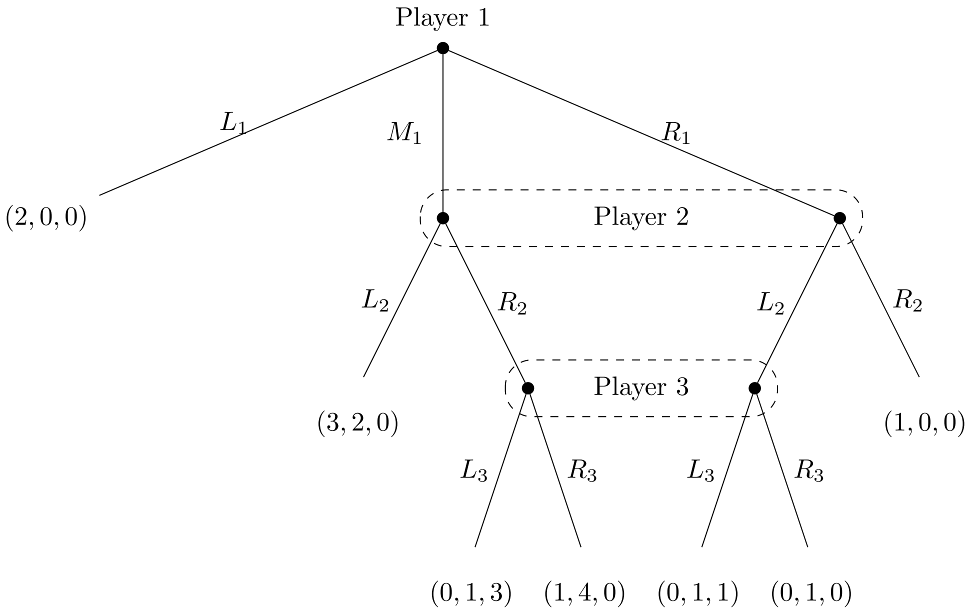
Definition 1. *The extensive-form game in Example 1 consists of the following:*

1. *Players:* $\mathcal{N} = \{1, 2\}$.

2. *Histories:* $\mathcal{H} = \{\overbrace{\varnothing}^{\text{initial}}, U, D, \overbrace{UL, UR, DL, DR}^{\text{terminal histories}}\}$ *is the set of all histories;* $\mathcal{Z} = \{UL, UR, DL, DR\}$ *is the set of terminal histories.*
3. *Player function* $\mathscr{P} : \mathcal{H} \setminus \mathcal{Z} \to \mathcal{N}$*, which maps non-terminal histories to the set of players:* $\mathscr{P}(\varnothing) = 1$ *and* $\mathscr{P}(U) = \mathscr{P}(D) = 2$.
4. *Payoff functions* $u_i : \mathcal{Z} \to \mathbb{R}$*, which map terminal histories to payoffs for each player* $i \in \mathcal{N}$ *(see the game tree for the payoffs).*

1.2 Imperfect information without exogenous uncertainty

Example 2. *Consider the following extensive-form game:*



To formally define the extensive-form game in Example 2, we also need to specify the set of players, the set of histories of play, specifying which player moves at each non-terminal history, and the payoffs achieved by the players at each terminal history. Additionally, we need to specify *information sets*, which are subsets of histories that the players cannot distinguish. The formal definition of Example 2 is as follows:

Definition 2. *The extensive-form game in Example 2 consists of the following:*

1. *Players:* $\mathcal{N} = \{1, 2, 3\}$.
2. *Histories: the set of all histories is given by*

$$\mathcal{H} = \{\varnothing, L_1, M_1, R_1, M_1L_2, M_1R_2, R_1L_2, R_1R_2, M_1R_2L_3, M_1R_2R_3, R_1L_2L_3, R_1L_2R_3\}.$$

The set of terminal histories is given by

$$\mathcal{Z} = \{L_1, M_1L_2, R_1R_2, M_1R_2L_3, M_1R_2R_3, R_1L_2L_3, R_1L_2R_3\}.$$

3. *Player function* $\mathscr{P} : \mathcal{H} \setminus \mathcal{Z} \to \mathcal{N}$*, which maps non-terminal histories to the set of players:* $\mathscr{P}(\varnothing) = 1$*;* $\mathscr{P}(M_1) = \mathscr{P}(R_1) = 2$ *and* $\mathscr{P}(M_1R_2) = \mathscr{P}(R_1L_2) = 3$*.*
4. *Collections of information sets for each player:* $\mathcal{I}_1 = \{\{\varnothing\}\}$*,* $\mathcal{I}_2 = \{\{M_1, R_1\}\}$*, and* $\mathcal{I}_3 = \{\{M_1R_2, R_1L_2\}\}$*.*
5. *Payoff functions* $u_i : \mathcal{Z} \to \mathbb{R}$*, which map terminal histories to payoffs for each player* $i \in \mathcal{N}$ *(see the game tree for the payoffs).*

Remark 1. *We can also define information sets for games of perfect information. In games of perfect information, each information set consists of a single history. In Example 1 we have* $\mathcal{I}_1 = \{\{\varnothing\}\}$ *and* $\mathcal{I}_2 = \{\{U\}, \{D\}\}$*.*

2 Strategies and equilibria

Definition 3. *A pure strategy in an extensive-form game is a function that maps information sets to actions, i.e.* $\sigma_i : I_i \mapsto \sigma_i(I_i) \in A(I_i)$*, where* $A(I_i)$ *is the set of actions available to player* i *in information set* I_i*.*

In Example 1, the set of pure strategies of player 1 is $S_i = \{U, D\}$, the set of pure strategies of player 2 is $S_2 = \{LL, LR, RL, RR\}$. In Example 2, the set of pure strategies of player 1 is $S_1 = \{L_1, M_1, R_1\}$, the set of pure strategies of player 2 is $S_2 = \{L_2, R_2\}$, and the set of pure strategies of player 3 is $S_3 = \{L_3, R_3\}$.

2.1 Strategic form and Nash equilibria

Having defined the strategies, we can rewrite Examples 1 and 2 in strategic form and look for their Nash equilibria in pure strategies. The strategic form of the game in Example 1 is given by the following payoff table:

	LL	LR	RL	RR
U	$2, 1$	$2, 1$	$0, 0$	$0, 0$
D	$-1, 1$	$3, 2$	$-1, 1$	$3, 2$

(U, LL), (D, LR), and (D, RR) are pure Nash equilibria of Example 1.

The strategic form of the game in Example 2 is given by the following payoff table:

	L_1		M_1		R_1	
	L_3	R_3	L_3	R_3	L_3	R_3
L_2	$2, 0, 0$	$2, 0, 0$	$3, 2, 0$	$3, 2, 0$	$0, 1, 1$	$0, 1, 0$
R_2	$2, 0, 0$	$2, 0, 0$	$0, 1, 3$	$1, 4, 0$	$1, 0, 0$	$1, 0, 0$

(L_1, R_2, L_3), (L_1, R_2, R_3), and (M_1, L_2, L_3) are pure Nash equilibria of Example 2.

2.2 Subgame-perfect Nash equilibria

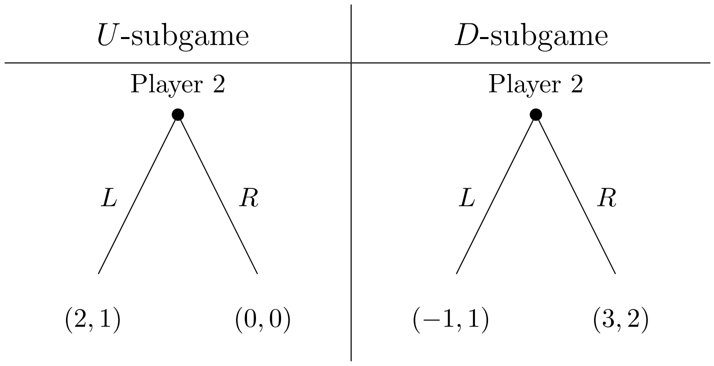
We are motivated by the fact that not all Nash equilibria are plausible predictions of the actual play in extensive-form games. Indeed, consider Example 1: if player 1 has played U, it does not make sense for player 2 to play R since L gives player 2 a higher payoff, likewise if player 1 has played D, it does not make sense for player 2 to play L because R gives player 2 a higher payoff. Hence the only plausible equilibrium here is (D, LR). To formalize this argument, we introduce the notions of a subgame and a subgame-perfect Nash equilibrium:

Definition 4 (Subgame). *Subgame is a part of an extensive-form game that satisfies the following conditions:*

1. *The initial node of the subgame is the only node in its information set.*
2. *If a node belongs to the subgame, then so do its successors.*
3. *If a node from an information set belongs to the subgame, then so do all nodes in this information set.*

Definition 5 (Subgame-perfect Nash equilibrium). *A strategy profile is a subgame-perfect Nash equilibrium if it induces a Nash equilibrium in every subgame.*

In Example 1, there are 3 subgames: the whole game and 2 proper subgames:



(D, LR) is the only subgame-perfect Nash equilibrium in Example 1.

2.3 Weak perfect Bayesian equilibria

Consider now Example 2. The only subgame in Example 2 is the whole game, hence all of its Nash equilibria are subgame-perfect. We will however argue that not all of them are plausible predictions of the actual play. To see that, let us introduce a *belief system*: let us suppose that player 3 believes that she is at history M_1R_2 with probability μ_3 and at history R_1L_2 with probability $1 - \mu_3$. The expected payoffs of player 3 are then given by:

$$L_3: \; 3\mu_3 + 1(1 - \mu_3) = 2\mu_3 + 1,$$
$$R_3: \; 0\mu_3 + 0(1 - \mu_3) = 0,$$

therefore for any belief μ_3 it is optimal for player 3 to choose L_3.

Suppose further that player 2 believes that she is at history M_1 with probability μ_2 and at history R_1 with probability $1-\mu_2$. Player 2 knows that player 3 will choose L_3, hence the expected payoffs of player 2 are given by:

$$L_2: \; 2\mu_2 + 1(1 - \mu_2) = \mu_2 + 1,$$
$$R_2: \; 1\mu_2 + 0(1 - \mu_2) = \mu_2,$$

therefore for any belief μ_2 it is optimal for player 2 to choose L_2.

Knowing that players 2 and 3 will choose L_2 and L_3 respectively, player 1 will find it optimal to choose M_1, and hence (M_1, L_2, L_3) is the only plausible equilibrium. To formalize this idea, we introduce the notion of a weak perfect Bayesian equilibrium.

Definition 6 (**Weak perfect Bayesian equilibrium**). *A strategy profile σ and a belief system μ is a weak perfect Bayesian equilibrium if*

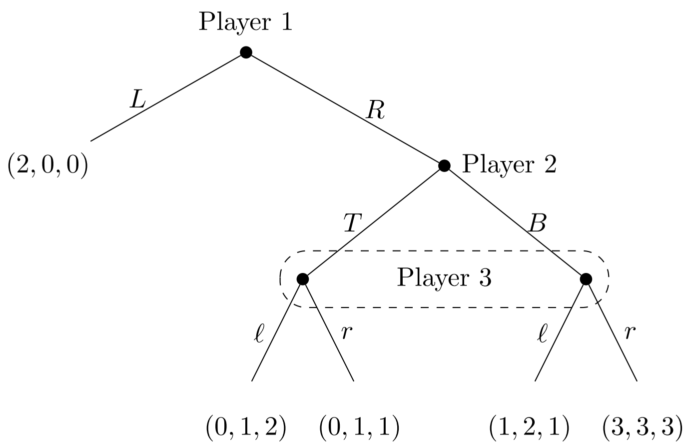
1. *σ is sequentially rational given μ, i.e. at every information set each player maximizes her expected utility given her beliefs.*
2. *For every information set reached with positive probability given σ, the beliefs at this information set are derived via Bayes rule.*

In Example 2, if the players play $\left(M_1, L_2, L_3\right)$, the information set of player 2 is reached with probability 1, hence we must have $\mu_2^* = 1$, but the information set of player 3 is reached with probability 0, hence we can choose any μ_3^* that makes it optimal for player 3 to play L_3. In our example, any $\mu_3^* \in [0,1]$ happens to work. Since we have established above that playing (M_1, L_2, L_3) is sequentially rational given $\mu_2^* = 1$ and $\mu_3^* \in [0,1]$, we conclude that $\left((M_1, L_2, L_3); \mu_2^* = 1, \mu_3^* \in [0,1]\right)$ are weak perfect Bayesian equilibria with (M_1, L_2, L_3) being the unique weak perfect Bayesian equilibrium strategy profile.

2.4 Some weak perfect Bayesian equilibria are *not* subgame-perfect

Example 2 illustrates how some subgame-perfect equilibria are not weak perfect Bayesian. In the next example, we will show that some weak perfect Bayesian equilibria are not subgame-perfect.

Example 3. *Consider the following extensive-form game:*



The formal definition of the game in Example 3 is as follows:

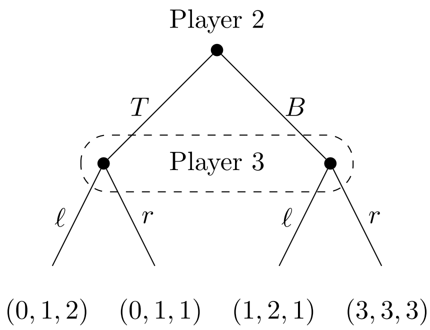
Definition 7. *The extensive-form game in Example 3 consists of the following:*

1. *Players:* $\mathcal{N} = \{1, 2, 3\}$.
2. *Histories:* $\mathcal{H} = \{\emptyset, L, R, RT, RB, RT\ell, RTr, RB\ell, RBr\}$,

   Terminal histories: $\mathcal{Z} = \{L, RT\ell, RTr, RB\ell, RBr\}$.
3. *Player function:* $\mathscr{P}(\emptyset) = 1$, $\mathscr{P}(R) = 2$, $\mathscr{P}(RT) = \mathscr{P}(RB) = 3$.
4. *Collections of information sets:* $\mathcal{I}_1 = \{\{\emptyset\}\}$, $\mathcal{I}_2 = \{\{R\}\}$, *and* $\mathcal{I}_3 = \{\{RT, RB\}\}$.
5. *Payoff functions* $u_i : \mathcal{Z} \to \mathbb{R}$ *(see the game tree for the payoffs).*

2.4.1 Subgame-perfect Nash equilibria

Let us first look at the subgame-perfect Nash equilibria of Example 3. This game has 2 subgames: the whole game and the following proper subgame:



The strategic form of this proper subgame is given by

	ℓ	r
T	0,1,2	0,1,1
B	1,2,1	3,3,3

The unique Nash equilibrium in this subgame is (B, r), and therefore (R, B, r) is the unique subgame-perfect Nash equilibrium in the whole game.

2.4.2 Weak perfect Bayesian equilibria

To find weak perfect Bayesian equilibria, let us assume that Player 3 believes that she is at history RT with probability μ and at history RB with probability $1 - \mu$. The expected payoffs of Player 3 are then given by:

$$\ell: \ 2\mu + 1(1-\mu) = \mu + 1,$$

$$r: \ 1\mu + 3(1-\mu) = 3 - 2\mu.$$

Player 3 will choose ℓ when $\mu + 1 \geq 3 - 2\mu$ or $\mu \geq \frac{2}{3}$, and will choose r otherwise. We therefore distinguish two cases.

Case 1: Player 3 chooses ℓ, then $\mu \geq \frac{2}{3}$. Knowing that, player 2 will choose B. Player 1 will then choose L. The information set of Player 3 is reached with probability 0 given (L, B, ℓ), hence $\big((L, B, \ell); \mu^* \in [\frac{2}{3}, 1]\big)$ are all weak perfect Bayesian equilibria.

Case 2: Player 3 chooses r, then $\mu \leq \frac{2}{3}$. Knowing that, player 2 will choose B. Player 1 will then choose R. The information set of player 3 is reached with probability 1 given (R, B, r), hence $\mu^* = 0$ and $\big((R, B, r); \mu^* = 0\big)$ is a weak perfect Bayesian equilibrium.