Game Theory, Spring 2024
Lecture # 8

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1 Repeated games: introduction

Example 1. *Consider the following prisoner's dilemma:*

	c	d
c	$5, 5$	$1, 6$
d	$6, 1$	$2, 2$

We know that if the game in Example 1 is played once, the players will never cooperate as (d, d) is the only Nash equilibrium. In our study of repeated games we are primarily interested in understanding whether (and to what extent) repetition of an interaction helps to sustain cooperative outcomes.

The basic building block of a repeated game is its *stage game*. We are going to study stage games that are strategic-form games with complete information (Example 1 is one of those), characterized by:

1. Players $i \in \mathcal{I} = \{1, \dots, I\}$,
2. Actions $a_i \in A_i$ for each player $i \in \mathcal{I}$,
3. Payoffs $u_i(a_i, a_{-i})$ for each player $i \in \mathcal{I}$.

We are interested in T-times repetitions of stage games, where T could be *finite* or *infinite*. Let $a^t = (a_1^t, \dots, a_I^t)$ be the action profile played in period t. The utility of player i in a T-times repeated game is given by:

$$u_i(a^0) + \delta u_i(a^1) + \delta^2 u_i(a^2) + \cdots = \sum_{t=0}^{T} \delta^t u_i(a^t),$$

where δ is a *discount factor*. If $T = \infty$, we assume $\delta \in (0, 1)$; if $T < \infty$, we might include 1 and assume $\delta \in (0, 1]$. When convenient, we normalize the utility as follows:

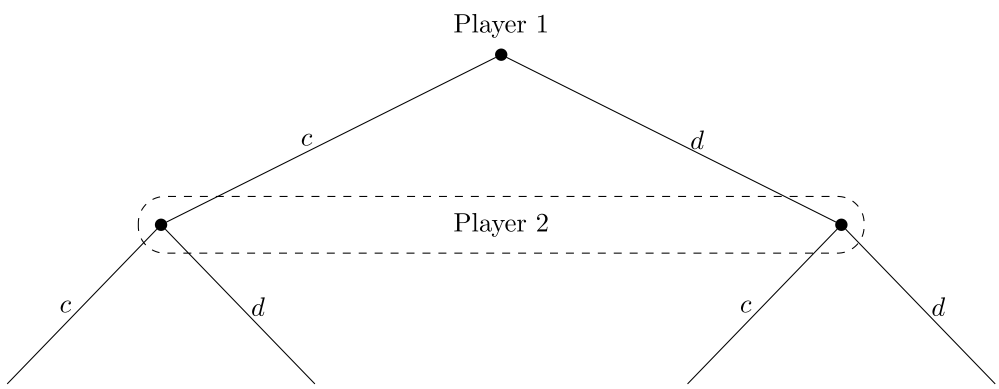
$$(1 - \delta) \sum_{t=0}^{T} \delta^t u_i(a^t).$$

We are interested in subgame-perfect Nash equilibria of repeated games.

2 Finitely repeated games

Example 2. *Consider the prisoner's dilemma from Example 1 played twice.*

We can treat Example 2 as an extensive-form game. Its game tree is given by:



	c	d
c	$5+\delta 5, 5+\delta 5$	$5+\delta, 5+\delta 6$
d	$5+\delta 6, 5+\delta$	$5+\delta 2, 5+\delta 2$

	c	d
c	$1+\delta 5, 6+\delta 5$	$1+\delta, 6+\delta 6$
d	$1+\delta 6, 6+\delta$	$1+\delta 2, 6+\delta 2$

	c	d
c	$6+\delta 5, 1+\delta 5$	$6+\delta, 1+\delta 6$
d	$6+\delta 6, 1+\delta$	$6+\delta 2, 1+\delta 2$

	c	d
c	$2+\delta 5, 2+\delta 5$	$2+\delta, 2+\delta 6$
d	$2+\delta 6, 2+\delta$	$2+\delta 2, 2+\delta 2$

There are four proper subgames in Example 2. The unique Nash equilibrium in every proper subgame is (d, d), hence (d, d) will be the unique subgame-perfect equilibrium outcome in the initial round as well, and coopoeration is impossible to sustain anywhere despite the repetition.

Example 3. *Consider the following stage game:*

	c	k	d
c	$5,5$	$0,0$	$1,6$
k	$0,0$	$4,4$	$0,0$
d	$6,1$	$0,0$	$2,2$

The stage game in Example 3 has two Nash equilibria in pure strategies: (k,k) and (d,d). Suppose this stage game is played twice and $\delta = 1$. We will show that it is possible to sustain cooperation in the initial round (i.e. play (c,c) in the initial round of a subgame-perfect Nash equilibrium) despite (c,c) not being a Nash equilibrium of the stage game.

Claim 1. *Suppose σ^* is the following strategy:*

1. *in the initial period, play c;*
2. *in the last period,*
   - *play k, if both players have played c in the initial period,*
   - *play d, otherwise.*

(σ^*, σ^*) is a subgame-perfect Nash equilibrium of the repeated game.

Proof. Clearly, the play in every proper subgame is a Nash equilibrium. The choices in the initial period are then given by:

	c	k	d
c	$9,9$	$2,2$	$3,8$
k	$2,2$	$6,6$	$2,2$
d	$8,3$	$2,2$	$4,4$

Since (c,c) is a Nash equilibrium of the above game, (σ^*, σ^*) is a subgame-perfect Nash equilibrium of the repeated game. □

3 Infinitely repeated games

Example 4. *Consider the infinitely repeated version of Example 1.*

We are going to show that it is possible to sustain cooperation (i.e. play (c,c)) in every period in Example 4 via grim-trigger strategies:

Definition 1 (**Grim-trigger strategy**). *σ^* is a grim-trigger strategy if it prescribes the following:*

1. *play c in the initial period,*
2. *in every other period,*
    - *play c, if everybody has played c before,*
    - *play d, otherwise.*

If both players play σ^*, then player i's payoff:

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t 5 = (1-\delta)\frac{5}{1-\delta} = 5.$$

We have to consider two kinds of histories:

1. If anyone has played d before, σ^* prescribes to play d. Since (d,d) is a Nash equilibrium in the stage game, playing (d,d) forever is a subgame-perfect Nash equilibrium after those history.
2. If nobody has played d before, σ^* prescribes to play c. Let us consider a *one-shot deviation* from σ^*, i.e. another strategy σ' that prescribes to play c in the current period and then returns to playing σ^* from next period on. σ' is not profitable as long as:
$$\underbrace{5}_{\text{payoff of } \sigma^*} \geq \underbrace{(1-\delta)6+\delta 2}_{\text{payoff of } \sigma'},$$
i.e. whenever $\delta \in \left[\frac{1}{4}, 1\right)$.

Hence for all $\delta \in \left[\frac{1}{4}, 1\right)$ the strategy profile (σ^*, σ^*) is a subgame-perfect equilibrium of the repeated prisoner's dilemma.

Grim-trigger strategies are reasonably simple, and therefore it is almost immediately clear that considering one-shot deviations from them is without loss of generality. With other, more complicated strategies, it might be less clear. It is however true in general. To establish that, we will introduce the following formal definitions:

Definition 2 (**Histories**). *A period-t history is a sequence of action profiles up to time t, i.e. $h^t = (a^0, \ldots, a^t) \in \mathcal{H}$.*

Definition 3 (**Pure strategy**)**.** *A pure strategy of player i is a function $\sigma_i : \mathcal{H} \to A_i$.*

Definition 4 (**One-shot deviation**)**.** *A one-shot deviation for player i from strategy σ_i is a strategy $\hat{\sigma}_i \neq \sigma_i$ such that there is a history $\tilde{h}^t$ such that for all $h^t \neq \tilde{h}^t$ we have $\sigma_i(h^t) = \hat{\sigma}_i(h^t)$.*

We establish the following proposition:

Proposition 1 (**The one-shot deviation principle**)**.** *A strategy profile σ is a subgame-perfect Nash equilibrium iff there are no profitable one-shot deviations.*

Proof. We will discuss the proof during Lecture #9. □